Solution 6

1. Let $f : E \to Y$ be a uniformly continuous map where $E \subset X$ and X, Y are metric spaces. Suppose that Y is complete. Show that there exists a uniformly continuous map F from $\overline{E}$ to Y satisfying $F = f$ in E. In other words, f can be extended to the closure of E preserving uniform continuity.

   Solution. Let $x \in \partial E$ and $\{x_n\}, x_n \in E, x_n \to x$. By uniform continuity, one readily sees that $\{f(x_n)\}$ is a Cauchy sequence in Y. As Y is complete, $f(x_n) \to z$ for some $z \in Y$. We define $F(x) = z$ to get an extension of f from E to $\overline{E}$. It is easy to check that this definition is independent of the sequence $\{x_n|\}$. Moreover, F is uniformly continuous on $\overline{E}$.

   Note. A contraction is, in particular, uniformly continuous. This property was used in Remark 3.1(a).

2. Let $A = \{a_{ij}\}$ be an $n \times n$ matrix. Show that

$$|Ax| \leq \sqrt{\sum_{i,j} a_{ij}^2}\ |x|.$$

   Solution. Let $y = Ax$. We have

$$y_i = \sum_j a_{ij} x_j, \quad i = 1, \cdots, n\ .$$

   By Cauchy-Schwarz Inequality,

$$|y_i| \leq \sqrt{\sum_j a_{ij}^2} \sqrt{\sum_j x_j^2}\ .$$

   Taking square,

$$y_i^2 \leq \sum_j a_{ij}^2 \sum_j x_j^2\ .$$

   Summing over i,

$$\sum_i y_i^2 \leq \sum_{i,j} a_{ij}^2 \sum_j x_j^2\ ,$$

   and the result follows by taking root.

   Note. This result was used in the proof of Proposition 3.5.

3. Can you solve the system of equations

$$x + y^4 = 0, \quad y - x^2 = 0.015\ ?$$

   Solution. Here we work on $\mathbb{R}^2$ and $\Phi(x, y) = (x, y) + \Psi(x, y)$ where $\Psi(x, y) = (-y^4, x^2)$. We have $\Phi(0, 0) = (0, 0)$ and want to solve $\Phi(x_1, x_2) = (0, 0.015)$. In the following points in $\mathbb{R}^2$ are denoted by $p = (x_1, y_1), q = (x_2, y_2)$, etc.

$$\begin{aligned}
\|\Psi(p) - \Psi(q)\|_2 &= \|(-y_1^4 + y_2^4, x_1^2 - x_2^2)\|_2 \\
&= \|((y_1^2 + y_2^2)(y_1 + y_2)(y_2 - y_1), (x_1 + x_2)(x_1 - x_2)\|_2 \\
&\leq \sqrt{(2r^2 \times 2r)^2 + (2r)^2}\|p - q\|_2 \\
&= 2r(1 + 4r^2)\|p - q\|_2\ .
\end{aligned}$$

(We have used $|x_1 - x_2|, |y_1 - y_2| \le \|p - q\|_2$.) Hence by taking $r = 1/4, \gamma = 5/8$ and $R = 3/24 = 0.125$. As $0.015 < 0.125$, the system is solvable.

4. Can you solve the system of equations

$$x + y - x^2 = 0, \quad x - y + xy \sin y = -0.002 \ ?$$

Hint: Put the system in the form $x + \cdots = 0, \quad y + \cdots = 0$, first.

Solution. First we rewrite the system in the form of $I + \Psi$. Indeed, by adding up and subtracting the equations, we see that the system is equivalent to

$$x + (-x^2 + xy \sin y)/2 = -0.001, \quad y + (-x^2 - xy \sin y)/2 = 0.001 \ .$$

Now we can take

$$\Psi(x, y) = \frac{1}{2}(-x^2 + xy \sin y, -x^2 - xy \sin y) \ ,$$

and proceed as in the those examples in Notes.

5. Let $A = (a_{ij})$ be an $n \times n$ matrix. Show that the matrix $I + A$ is invertible if $\sum_{i,j} a_{ij}^2 < 1$. Give an example showing that $I + A$ could become singular when $\sum_{i,j} a_{ij}^2 = 1$.

Solution. Let $\Phi(x) = Ix + Ax$ so that $\Psi(x) = Ax$ for $x \in \mathbb{R}^n$. By the previous problem,

$$|\Psi(x_1) - \Psi(x_2)| = |A(x_1 - x_2)| \le \sqrt{\sum_{i,j} a_{ij}^2} \ |x| \ .$$

Take $\gamma = \sqrt{\sum_{i,j} a_{ij}^2} < 1$. Ψ is a contraction and there is only one root of the equation $\Phi(x) = 0$ in the ball $B_r(0)$. However, since we already know $\Phi(0) = 0$, 0 is the unique root. Now, we claim that $I + A$ is non-singular, for there is some $z \in \mathbb{R}^n$ satisfying $(I + A)z = 0$, we can find a small number α such that $\alpha z \in B_r(0)$. By what we have just shown, $\alpha z = 0$ so $z = 0$, that is, $I + A$ is non-singular and thus invertible.

The sharpness of the condition $\sum a_{ij}^2 < 1$ can be seen from considering the 2×2-matrix A where all $a_{ij} = 0$ except $a_{22} = -1$.

Note. See how linearity plays its role in the proof.

6. Consider the iteration

$$x_{n+1} = \alpha x_n(1 - x_n), \ x_1 \in [0, 1] \ .$$

Find

(a) The range of α so that $\{x_n\}$ remains in $[0, 1]$.

(b) The range of α so that the iteration has a unique fixed point 0 in $[0, 1]$.

(c) Show that for $\alpha \in [0, 1]$ the fixed point 0 is attracting in the sense: $x_n \to 0$ whenever $x_0 \in [0, 1]$.

Solution. Let $Tx = \alpha x(1 - x)$. The max of T attains at $1/2$ so the maximal value is $\alpha/4$. Therefore, the range of α is $[0, 4]$ so that T maps $[0, 1]$ to itself. Next, 0 is always a fixed point of T. To show that is it, we set $x = \alpha x(1 - x)$ and solve for x and get

$x = (\alpha - 1)/\alpha$. So there is no other fixed point if $\alpha \in [0, 1]$. Finally, it is clear that T becomes a contraction when $\alpha \in [0, 1)$, so the sequence $\{x_n\}$ with $x_0 \in [0, 1]$, $x_n = T^n x_0$, always tends to 0 as $n \to \infty$. Although T is not a contraction when $\alpha = 1$, one can still use elementary mean (that is, $\{x_n\}$ is always decreasing,) to show that 0 is an attracting fixed point.